Optimization Models and Methods

MIT 15.053, Spring 2025, Project 2
Kuhn Poker Project

Due: April 28, 2025 at 10 AM EST

Project rules:

1. Each group should submit a single "solution set."
2. You may use ChatGPT, GPT-4, or another LLM only in the following ways: (1) as reference, such as asking them to explain definitions or concepts; (2) to help create and debug Julia/JuMP code, and (3) to help describe your optimal linear programming solution for the 4-card version of Kuhn Poker.
3. Do not use other resources on the web to answer questions.
4. Late assignments will *not* be accepted. There are no "flex days" for projects.
5. The solutions should be submitted to Gradescope and Canvas prior to 10 AM on the day it is due. The PDF of your written answers should be submitted to Gradescope (Photos are also acceptable.) If the scans are not sufficiently readable, we may request that they be rescanned. Please also submit any code files used to Canvas.
6. Please include screenshots of any Excel spreadsheets/Julia code used in solution write-ups (with the corresponding question) to allow for easier grading by the TAs.

Problem 1. Simplified Kuhn Poker (15 points)

In expressing Simplified Kuhn Poker as a 2-person 0-sum game, each pure strategy is a description of what the player would do under each circumstance. There are three possible scenarios. Player 1 can receive a 1, a 2, or a 3. In each scenario, Player 1 can either check or bet. This leads to 8 different pure strategies. Each strategy is a triple where each component is a card value and a B (bet) or a C (check). For example, the strategy [1C, 2C, 3B] means that Player 1 would check with a 1, check with a 2 and bet with a 3. The 8 possible strategies are: (1) [1C, 2C, 3C], (2) [1C, 2C, 3B], (3) [1C, 2B, 3C], (4) [1C, 2B, 3B], (5) [1B, 2C, 3C], (6) [1B, 2C, 3B], (7) [1B, 2B, 3C], and (8) [1B, 2B, 3B]. (Your group may use a different notation if you want.)

Player 2 also has three scenarios, according as she receives a 1, 2, or 3. If Player 1 checks, then Player 2 must also check. If Player 1 bets, then Player 2 has two possible choices: Call (Ca) or fold (F). We represent strategies for Player 2 as triples. For example, the strategy [1F, 2F, 3Ca] means if Player 1 bets, then Player 2 would fold with a 1, fold with a 2 and call with a 3. This leads to 8 different pure strategies. (1) [1Ca, 2Ca, 3Ca], (2) [1Ca, 2Ca, 3F], (3) [1Ca, 2F, 3Ca], (4) [1Ca, 2F, 3F], (5) [1F, 2Ca, 3Ca], (6) [1F, 2Ca, 3F], (7) [1F, 2F, 3Ca], and (8) [1F, 2F, 3F].

(a) (1 point.) Can this game be expressed as a two-person, zero-sum game? (HINT: The answer is yes.) For convenience, refer to Player 1 as the row player and refer to Player 2 as the column player.

(b) (3 points.) Which of the pure strategies for Player 2 are dominated? For each dominated strategy, say which of the other strategies dominates it.

(c) (3 points.) Suppose that one first eliminates the dominated pure strategies for Player 2. For the resulting payoff matrix, which pure strategies of Player 1 are dominated? For each dominated strategy, say which of the other strategies dominates it. (For example, if Player 1 bets, and if Player 2 has a 3 then it is better for Player 2 to call rather than fold.)

(d) (3 points.) After eliminating all dominated columns (Player 2's strategies) and then eliminating all dominated rows (Player 1's strategies), write the payoff matrix. What are the remaining (pure) strategies for each player? What is the payoff matrix?

(e) (5 points.)What is the optimal mixed strategy for Player 1 and Player 2? What is the maximin expected payoff to Player 1? (You should state how you found the optimal solution. You may use any technique that you want.)

Problem 2. Kuhn Poker (24 points)

Kuhn Poker differs from Simplified Kuhn Poker in that Player 2 is permitted to bet if Player 1 checks. If Player 2 bets, then Player 1 either calls or folds.

Let's look at the game from Player 1's perspective. This is illustrated in Figure 1. First, Player 1 sees whether she has a 1, 2, or 3. Then she needs to make a decision: bet or check. If she makes the decision to bet, then Player 2 either calls or folds. If Player 2 calls, then the cards are revealed and the winner of the hand is determined. (This last step is not outlined in the figure.) If Player 1 checks, then Player 2 can either check or bet. If Player 2 bets, then Player 1 faces another decision: call or fold.

There are six different decision points in Figure 1. A pure strategy for Player 1 is a specification of the decision to make at each decision point. This is illustrated in Figure 2. Notice that although there are six decision points, three of them are conditioned on a her checking first. In this decision, she bet with a 3, and so there were only 5 decisions.

When expressing the rows of the payoff matrix, it is useful to have a "shorthand" way of describing all of the strategies. Here we propose one possible shorthand. To describe the strategies, we use the following notation. Let F denote a strategy in which Player 1 first checks and then folds if Player 2

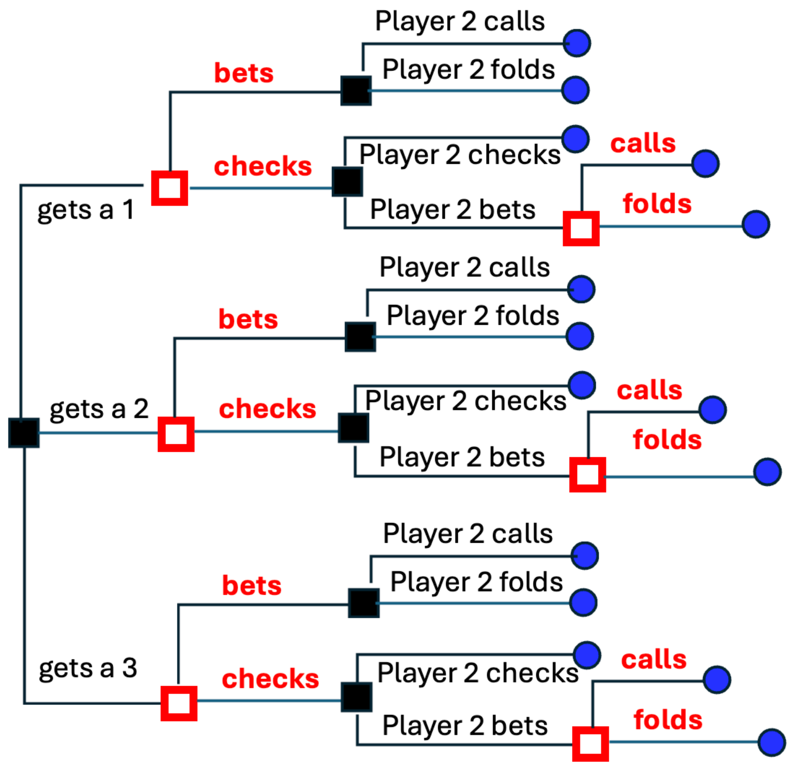

Figure 1: A tree of Kuhn Poker for player 1. The decisions that Player 1 needs to make are in red and are boldface.

bets. Let Ca denote a strategy in which Player 1 first checks and then calls if Player 2 bets. For example, the strategy [1F, 2Ca, 3B] refers to the case in which Player 1 (i) checks with a 1 and then folds if Player 2 bets, (ii) checks with a 2 and then calls if Player 2 bets, and (iii) bets with a 3.

Player 2 also has more strategies. To describe the strategies, we use the following notation. Let CaC denote a strategy in which Player 2 calls if Player 1 bets and checks if Player 1 checks. Let CaB denote a strategy in which Player 2 calls if Player 1 bets and bets if Player 2 calls. Let FC denote a strategy in which Player 2 folds if Player 1 bets and checks if Player 1 checks. Let FB denote a strategy in which Player 1 folds if Player 1 bets, and bets if Player 1 folds. As before, we represent a strategy by a triple. For example, [1FB, 2CaC, 3CaB] represents the strategy FB if Player 2 receives a 1, strategy CaC if Player 2 receives a 2, and CaB if Player 2 receives a 3.

Hint. You may find it helpful if you write a function that provides the payoff to Player 1 (possibly positive or negative) based on the following four inputs: (i) the card that Player 1 gets, (ii) the card that Player 2 gets, (iii) the strategy for Player 1 for his card (i.e, Ca or F or B), and (iv) the strategy for Player 2 for her card (i.e., CaC, or CaB, or FC or FB). This function can be used as a subroutine to obtain the entries to the payoff matrix.

(a) (1 point.) Describe your notation for strategies for Players 1 and 2. (If it is the same as above, say so.) How many different pure strategies are there for Player 1? How many pure strategies are there for Player 2?

(b) (4 points.) (*Early detection of domination.*) Prior to determining values in the payoff matrix, determine some of the strategies for Player 2 that are dominated. For each dominated strategy, give the strategy that dominates it. (For example, if Player 2 has a 1 and if Player 1 bets, then

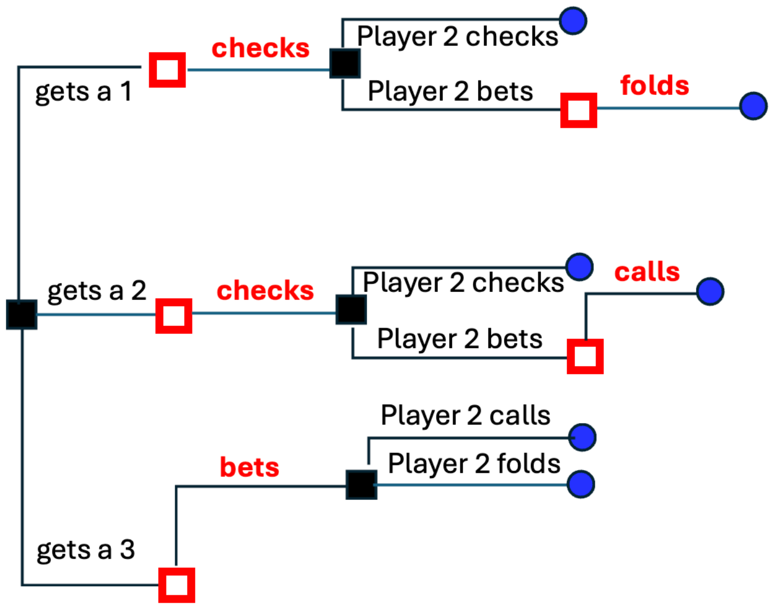

Figure 2: A pure strategy for Player 1. At every decision point for Player 1, exactly one decision is made.

it is better for Player 2 to fold rather than call.)

(c) (4 points) (*Early detection of domination.*) Prior to determining values in the payoff matrix, determine some of the strategies for Player 1 that are dominated. For each dominated strategy, give the strategy that dominates it. (You may use the fact that the dominated columns for Player 2 that you identified in the previous part have been eliminated.)

(d) (5 points.) Create a function in Julia that does the following: Given a strategy for Player 1 and a strategy for Player 2, it computes the expected payoff to Player 1.

Use the function to create the payoff matrix for the game. You may either use all of the strategies for Players 1 and 2 or you can restrict the payoff matrix to strategies that have not yet been eliminated.

(e) (8 points) Find the optimal mixed strategies for Player 1 and Player 2 by optimizing linear programs using Julia and JuMP. What are the optimal mixed strategies? What is the optimal expected payoff to Player 1?

(f) (2 points) Is the game fair? That is, is the maximin value equal to 0? If not, who has the advantage?

Problem 3. Kuhn Poker with larger bets. (15 points) This is the same game as Kuhn Poker with one exception. The bets are all \$3. That is, each Player puts an ante of \$1 into the pot. But subsequently, any bet is for \$3 rather than for \$1.

(a) (5 points.) Create a function in Julia that does the following: Given a strategy for Player 1 and a strategy for Player 2, it computes the expected payoff to Player 1.

Use the function to create the payoff matrix for the game. You may either use all of the strategies for Players 1 and 2 or you can restrict the payoff matrix to strategies that have not yet been eliminated.

(b) (8 points) Find the optimal mixed strategies for Player 1 and Player 2 by optimizing linear programs using Julia and JuMP. What are the optimal mixed strategies? What is the optimal expected payoff to Player 1?

(c) (2 points) Is the game fair? If not, who has the advantage?

Problem 4. Kuhn Poker with 4 cards. (18 points)

Now suppose that we begin Kuhn Poker (with \$1 bets) but with four cards rather than 3.

(a) (3 points) How does this change the description of the strategies for Player 1 and Player 2? How many pure strategies are there for Player 1 and Player 2?

(b) (5 points.) Create a function in Julia that does the following: Given a strategy for Player 1 and a strategy for Player 2, it computes the expected payoff to Player 1.

Use the function to create the payoff matrix for the game. Use all of the strategies for Players 1 and 2.

(c) (8 points) Find the optimal mixed strategies for Player 1 and Player 2 by optimizing linear programs using Julia and JuMP. What are the optimal mixed strategies? What is the optimal expected payoff to Player 1?

(d) (2 points) Is the game fair? If not, who has the advantage?

Problem 5. Bluffing in Kuhn Poker (28 points)

We now slightly shift gears to further evaluate how bluffing can influence the outcome of a Kuhn Poker game. Recall that bluffing in poker refers to a strategy where a player deliberately bets a larger amount than is indicated by the strength of their hand.

In the context of the 3-card games, we say that a bluff is a bet by a player who has been dealt a 1. In the previous exercises, you determined the optimal probability that each player should bluff. In this exercise, you will determine the "value of bluffing" in Kuhn Poker.

Before carrying out additional computations, discuss with other members of your group how you might assign a numerical value (or values) to the value of bluffing in Kuhn Poker. In addition, discuss with your group how large you expect the value(s) to be. Do you think that the value of bluffing will be higher if there are \$3 bets rather than \$1 bets?

(a) (3 points) Find the optimal payoff to Player 1 in simplified Kuhn Poker if Player 1 is not permitted to bluff. That is, Player 1 cannot bet if he has a 1, 2, or 3. What is the value of bluffing for Player 1? (This is the optimal payoff if bluffing is allowed minus the optimal payoff if bluffing is not allowed.)

(b) (3 points) Consider Kuhn Poker with \$1 bets. Compute the optimal payoff to Player 1 if Player 1 is permitted to bluff and Player 2 is not permitted to bluff. What is the increase in payoff to Player 1 if Player 2 is forbidden from bluffing?

(To prohibit bluffing for Player 2: for any strategy for Player 2 that includes bluffing, constrain the corresponding decision variable to be 0. A similar approach works for the next three parts of the problem.)

(c) (3 points) Consider again Kuhn Poker with \$1 bets. Compute the optimal payoff to Player 1 if Player 1 is not permitted to bluff and Player 2 is permitted to bluff. What is the decrease in payoff to Player 1 if Player 1 is forbidden from bluffing?

(d) (3 points) Consider Kuhn Poker with \$3 bets. Compute the optimal payoff to Player 1 if Player 1 is permitted to bluff and Player 2 is not permitted to bluff. What is the increase in payoff to Player 1 if Player 2 is forbidden from bluffing?

(e) (3 points) Consider again Kuhn Poker with \$3 bets. Compute the optimal payoff to Player 1 if Player 1 is not permitted to bluff and Player 2 is permitted to bluff. What is the decrease in payoff to Player 1 if Player 1 is forbidden from bluffing?

(f) (3 points) Based on the answers for the four different cases, summarize the advantage a player gets in Kuhn Poker and variants when they are permitted to bluff and the other player never bluffs.

(g) (10 points) Many poker players believe that the advantage of bluffing is primarily psychological. A player bluffs if the player believes that the opponents will fold. But bluffing in game theory involves no psychology. Write a paragraph or two explaining to these poker players where the value of bluffing comes from.

Do you anticipate that your explanation of the value of bluffing also applies to more complicated types of poker? Explain your reason(s).